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OPTIMAL DESIGN OF ELASTIC PLATES WITH A CONSTRAINT ON THE SLOPE OF THE THICKNESS FUNCTION

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Abstract—A constraint on the maximum absolute value of the slope of the thickness function is found to have a profound influence on the problem of optimal design of axisymmetric elastic plates by securing a unique, regular solution. It is also shown, that except for the case of axisymmetric deformation, the constraint is active everywhere and the optimal design is a plate, subdivided into annular regions of constant slope equal to the maximum value permitted.

1. INTRODUCTION

A notorious difficulty connected with the problem of optimal design of thin elastic plates is that unless specific precautions are taken, the set of admissible designs is open and does not contain the limiting case of maximum stiffness (minimum compliance) or maximum natural frequency.

There are two different ways of resolving this difficulty. One is to *extend* the design space to include some "generalized" structures, representing the limiting cases; the other is to *restrict* the design space to some closed subspace of the given space by introducing geometrical constraints.

In previous work on the optimization of thin elastic plates by the variational approach, both methods have been employed with the purpose of obtaining a well-posed problem.

Immediately, it seems to be more attractive to employ the first method, i.e. to close the design space by including the limiting designs.

In analogy with earlier work on columns and beams (see [1, 2]) this was done for the vibrating plate [3, 4], where a vanishing thickness at the simply supported boundary was allowed for. However, it was recognized that for the plate the solution was only a local optimum [4] and that there was no upper bound for the design variable (lowest frequency). This showed clearly that no well-posed problem could be obtained in the case of plates solely by extending the design space.

Consequently, in the next step geometrical constraints on the thickness (maximum and minimum thickness) of the plate were introduced [5], thereby restricting the design space. But as it appeared this did by no means close the design space and again the result proved that the limiting design did not belong to the design space. Therefore a "generalized" plate model was introduced [6] keeping the geometrical constraints on maximum and minimum thickness but allowing for an infinite number of infinitely thin "stiffeners", defined by a stiffener density. Thus the design space was restricted in one direction and extended in another. Now the problem appeared to become well-posed but the price was of course a rather exotic plate model.

From a practical point of view, the "stiffeners" cannot be made arbitrarily thin, and therefore it seems reasonable to introduce a constraint on the minimum width of a "stiffener". Even if it were possible to formulate and solve this as a pure plate problem (i.e. a problem in which a "stiffener" is merely treated as a—possibly discontinuous—variation of the thickness of a Kirchhoff plate) it could hardly be justified. Rather, one should then reformulate the problem as that of optimizing a stiffened plate of given total volume with uniform plate thickness and given characteristics of permissible stiffeners.

In the light of previous work it might be claimed that optimization of Kirchhoff plates leads to designs, for which the theory is not applicable. But that is not necessarily so, and in this paper we propose a formulation of the problem that will resolve this dilemma, namely by restricting the design space to plates of "slowly" varying thickness. In order to leave non-essentials out we shall limit our discussion to the optimal design for minimum compliance of axisymmetric plates of given volume. The thickness h is thus a function of the radius r only and we shall restrict ourselves to functions that are continuous, and satisfy the condition

$$|h'(r)| \leq s_{\max} \tag{1.1}$$

where prime denotes the derivative with respect to r and where $s_{\max}$ is a given non-negative number. It is remarkable that this constraint alone seems to be sufficient to ensure a well-posed problem. Furthermore, it has the attraction that the design may now be limited to such cases for which the Kirchhoff theory of thin plates is applicable.

As it turns out this constraint will prove to have a profound influence on the optimal design. In fact, we shall see that with the exception of the case of symmetrical loading, the condition (1.1) is *active everywhere* (implying a solution of the "bang-bang" type).

2. BASIC EQUATIONS

Let us consider a thin elastic homogeneous axisymmetric plate, bounded by the inner radius R_1 and the outer radius R_2, and let (r, ϕ) be polar coordinates on the middle-surface of the undeformed plate, concentric with the plate.

We shall assume that the plate is loaded by an external load

$$P(r, \phi) = p(r)\cos m\phi \tag{2.1}$$

where m is a non-negative integer. At the boundaries the plate is assumed to be free, simply supported, or clamped. Under these assumptions the deflection of the middle-surface is

$$W(r, \phi) = w(r)\cos m\phi \tag{2.2}$$

and the components of the bending tensor of the middle-surface are given by

$$K_{rr} = \kappa_{rr}(r)\cos m\phi;\; K_{r\phi} = \kappa_{r\phi}(r)\sin m\phi;\; K_{\phi\phi} = \kappa_{\phi\phi}(r)\cos m\phi$$

where

$$\begin{aligned} \kappa_{rr} &= w'' \\ \kappa_{r\phi} &= m(w' - w/r)/r \\ \kappa_{\phi\phi} &= (w' - m^2 w/r)/r. \end{aligned} \tag{2.3}$$

Following the assumptions of the Kirchhoff plate theory this leads to the differential equation

$$\begin{aligned} &\frac{d}{dr}\left\{\frac{d}{dr} D\left[r\frac{d^2w}{dr^2} + \nu\left(\frac{dw}{dr} - m^2\frac{w}{r}\right)\right]\right. \\ &\left. - D\left[\nu\frac{d^2w}{dr^2} + [2(1-\nu)m^2 + 1]\frac{1}{r}\frac{dw}{dr} - (3-2\nu)m^2\frac{w}{r}\right]\right\} \\ &- D\frac{m^2}{r}\left\{\nu\frac{d^2w}{dr^2} + (3-2\nu)\frac{1}{r}\frac{dw}{dr} - [2(1-\nu) + m^2]\frac{w}{r^2}\right\} = rp(r) \end{aligned} \tag{2.4}$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)} \tag{2.5}$$

and where E is Young's modulus, ν Poisson's ration and h the thickness of the plate. Equation (2.4) together with the appropriate boundary conditions determines the function $w(r)$.

The *compliance* Φ of the plate (for the load P) equals the work done by the external forces

$$\Phi = C\int_{R_1}^{R_2} p(r)w(r)r\,dr \tag{2.6}$$

where $C=\pi$ for $m=0$ and $C=\pi/2$ for $m>0$.

The problem can now be stated as to find a thickness function $h(r)$ from a given set of admissible functions that minimizes the compliance Φ. The load $p(r)$ and the number m is given.

Let us define the set of admissible functions. First, we restrict the set to continuous functions for which the first derivative is at least piecewise continuous in the interval $R_1 \le r \le R_2$. Secondly, we consider only plates of a given volume, so that h is subject to the volume restriction

$$\int_{R_1}^{R_2} h(r)r\,dr = 1. \tag{2.7}$$

Finally, we restrict the set to those functions h for which the slope is bounded by a given non-negative number s_{max},

$$|h'(r)| \le s_{max},\ R_1 \le r \le R_2. \tag{2.8}$$

These are the only constraints imposed on the design variable.

Since in our case of an elastic material, the work done by the external forces equals the strain energy, we have

$$C\int_{R_1}^{R_2} pwr\,dr = C\int_{R_1}^{R_2} DQ[\kappa]r\,dr$$

where Q is the following quadratic function of the components κ_{rr}, $\kappa_{\phi\phi}$, and $\kappa_{r\phi}$

$$Q[\kappa] = \kappa_{rr}^2 + \kappa_{\phi\phi}^2 + 2\nu\kappa_{rr}\kappa_{\phi\phi} + 2(1-\nu)\kappa_{r\phi}^2. \tag{2.9}$$

Consequently, we can write

$$\Phi = 2C\int_{R_1}^{R_2} pwr\,dr - C\int_{R_1}^{R_2} DQ[\kappa]r\,dr. \tag{2.10}$$

Let us assume that, for the optimal plate, the constraint (2.8) is active in a certain subdomain U_s of the total domain U ($R_1 \le r \le R_2$). Let δh be a small variation of the thickness function, such that $\delta h(r) \equiv 0$ whenever $r \in U_s$ and fulfilling the condition of no volume change,

$$\int_{R_1}^{R_2} \delta hr\,dr = 0 \tag{2.11}$$

but otherwise arbitrary.

When the thickness h is changed to $h+\delta h$, the deflection w changes to $w+\delta w$ and the total compliance Φ to $\Phi+\delta\Phi$. But the contribution to $\delta\Phi$ from δw according to (2.10) must be zero since w is the deflection in a state of equilibrium (principle of virtual work). Therefore, we have, also from (2.10)

$$\delta\Phi = -\frac{EC}{4(1-\nu^2)}\int_{R_1}^{R_2} h^2 Q[\kappa]\delta hr\,dr \tag{2.12}$$

For the optimal plate $\delta\Phi = 0$ and a comparison between (2.11) and (2.12) shows that

$$h^2 Q[\kappa] = \Lambda \text{ for } r \notin U_S \tag{2.13}$$

where Λ is a constant.

To determine the shape of the optimal plate we have to find the domain U_S in which the slope constraint is active and a thickness function which satisfies (2.13) in the remaining part U_0 of the total domain U.

3. THE CASE OF AXISYMMETRIC LOAD

In the axisymmetric case ($m = 0$) a considerable simplification is obtained when the annular region of the plate is assumed to be narrow. Let $R_2 - R_1$ be kept equal to 1 and let R_1 grow without limit. The differential equation (2.4) will then reduce to

$$\frac{d^2}{dx^2}\left(D\frac{d^2w}{dx^2}\right) = p(x) \tag{3.1}$$

where $x = r - R_1$. This is in fact the differential equation of an Euler–Bernoulli beam with bending stiffness D. Let us study the case of a simply supported plate (beam) with constant load $p(x) = q$.

The boundary conditions can be written as

$$w = h^3\frac{d^2w}{dx^2} = 0 \text{ for } x = 0$$

and (3.2)

$$\frac{dw}{dx} = \frac{d^3w}{dx^3} = 0 \text{ for } x = \frac{1}{2}$$

due to the symmetry of the solution. We shall confine the analysis to the region $0 \le x \le \frac{1}{2}$.

In this case, the constraint (2.8) is active in two regions close to the supports $x = 0$ and $x = 1$. We may therefore write

$$h(x) = \begin{cases} a + sx & 0 \le x \le x_0 \\ \kappa(x) & x_0 \le x \le \frac{1}{2} \end{cases} \tag{3.3}$$

where x_0 is unknown, but which shall be assumed fixed in the first step of the analysis. The function $\kappa(x)$ is determined from the condition of optimality (2.13), which in this case reduces to

$$\kappa^2\left(\frac{d^2w}{dx^2}\right)^2 = \Lambda \quad x_0 \le x \le \tfrac{1}{2}. \tag{3.4}$$

From (3.1), (3.2) and (3.4) we find

$$\kappa(x) = \lambda\sqrt{x(1-x)} \quad x_0 \le x \le \tfrac{1}{2} \tag{3.5}$$

where λ is undetermined as yet.

The thickness function must be continuous at $x = x_0$, i.e.

$$a + sx_0 = \lambda\sqrt{[x_0(1-x_0)]}$$

which determines λ. The constant a is then determined from the condition on the total volume,

$$\int_0^{x_0}(a+sx)\,dx + \int_{x_0}^{1/2}\kappa(x)\,dx = \frac{1}{2} \tag{3.6}$$

which yields the expression

$$a(x_0+J)=1-sx_0J \tag{3.7}$$

with

$$J=\frac{1}{2}+\frac{1}{\sqrt{[x_0(x-x_0)]}}\left(\frac{\pi}{8}-\frac{1}{2}\arcsin\sqrt{x_0}\right). \tag{3.8}$$

The compliance is determined from (2.9) and has the following form

$$\Phi=\left(\frac{q}{2}\right)^2\left\{\frac{x_0^4-3x_0^2-2\alpha x_0-(1+\alpha)(4x_0^3+18x_0^2+12\alpha^2x_0)}{2s(a+sx_0)^2}+\frac{x_0^2(1-x_0)^2(1-2ax_0-sx_0^2)}{2(a+sx_0)^4}+\frac{1+6\alpha(1+\alpha)}{s^3}\log\left(1+\frac{x_0}{\alpha}\right)\right\} \tag{3.9}$$

where $\alpha=a/s$.

The problem is now solved for a given value of $s=s_{max}$ by taking a starting value of x_0 and solving first a from (3.7), (3.8) and then $\Phi(x_0)$ from (3.9). The optimal value of x_0 is then determined by a bisectional procedure. Table 1 shows some results.

For large values† of s_{max} the domain of active constraint U_0, determined by x_0, consists of two narrow annular regions close to the boundaries. These regions widen when s_{max} decreases and meet each other at the center for $s_{max}=1$. For this value and smaller values of s_{max} the constraint is active everywhere and the design of the optimal plate becomes trivial. The slope $h'(x)$ of the optimal plate is discontinuous at x_0, which is seen from the last column in table 1. Of course, any assumption to the contrary would lead to a sub-optimal solution.

For values of $R_1/R_2<1$ the problem cannot be solved in closed form and we have to resort to a solution by successive iterations that yield a sequence of functions h_i.

With the simple case above in mind we assume that the constraint is active in two annular regions reaching inwards from the boundaries of the simply supported plate and determined by the inner radii ρ_1 and ρ_2, such that $R_1<\rho_1\le\rho_2<R_2$.

For a given thickness $h_i(r)$ the differential equation (2.4) is solved numerically by a Runge–Kutta procedure. This leads to a deflection function w_i and a corresponding compliance Φ_i. If we determine the next iteration h_{i+1} from the condition (2.13) in the following form,

$$h_{i+1}^2Q_i[\kappa]=\Lambda$$

Table 1. The limiting case $R_2/R_1\to 1$. The compliance is given as a fraction of the compliance of a plate of uniform thickness.

s_{max}	x_0	Φ/Φ_u	$\kappa'(x_0)$
0	—	1.0000	—
1	0.5000	0.8146	—
1.5	0.3970	0.7743	0.53
2	0.3157	0.7516	1.00
3	0.2022	0.7303	1.89
4	0.1352	0.7220	2.71
6	0.0697	0.7164	4.30
8	0.0415	0.7148	5.85
12	0.0193	0.7139	8.90
16	0.0110	0.7136	11.92
24	0.0050	0.7135	17.95
∞	0	0.7134	—

†In order to give the results a more general shape it is convenient to define the "slope" as a dimensionless multiple of $h_u/(R_2-R_1)$, where h_u is the thickness of a uniform plate of the given volume.

the sequence h_i will diverge. On the other hand, if we replace (2.13) by an equivalent expression in terms of moments and define h_{i+1} by

$$h_{i+1}^{-4}[M_{rr}^2 + M_{\phi\phi}^2 - 2\nu M_{rr} M_{\phi\phi} + 2(1+\nu) M_{r\phi}^2]_i = \Lambda \tag{3.10}$$

where

$$\begin{aligned} M_{rr} &= D(\kappa_{rr} + \nu\kappa_{\phi\phi}) \\ M_{\phi\phi} &= D(\kappa_{\phi\phi} + \nu\kappa_{rr}) \\ M_{r\phi} &= D(1-\nu)\kappa_{r\phi} \end{aligned} \tag{3.11}$$

all computed for h_i and the corresponding displacement w_i, we obtain a converging sequence h_i. In this procedure the constant Λ is determined from the conditions of continuity of $h(r)$ at ρ_1 and ρ_2 and the condition of given volume.

The result of this procedure yields the plate of minimum compliance $\Phi(\rho_1, \rho_2)$ for given values of ρ_1 and ρ_2. The solution of the problem, the optimal plate, is the one that yields the smallest value of $\Phi(\rho_1, \rho_2)$ for all ρ_1 and ρ_2. Thus

$$\Phi_{opt} = \underset{\rho_1, \rho_2}{\mathrm{Min}}\, \Phi(\rho_1, \rho_2)$$

and we find the minimizing values ρ_1 and ρ_2 by a bisectional method.

The following results were obtained.

With $R_1 = 100$, $R_2 = 101$ and $s_{max} = 6$ we found the compliance of the optimal plate, loaded by a uniformly distributed load, to be $\Phi/\Phi_u = 0.7166$ for $\rho_1 = 100.0675$ and $\rho_2 = 100.9305$. The result is thus quite close to the limiting case for $s_{max} = 6$. The shape of the cross-section is shown in Fig. 1, where one can notice the discontinuities of $h'(r)$ at ρ_1 and ρ_2.

When the ratio R_2/R_1 grows the solution becomes more asymmetrical and at some stage the character of the solution changes radically. For $R_2/R_1 = 5$ and $s_{max} = 6$ the cross-section of the optimal plate is shown in Fig. 2. The slope has changed sign in the inner domain of active constraint and has the same sign as in the outer domain. This can be ascribed to the beneficial influence of the torsional stiffness and the result is a considerably improved value of the compliance, $\Phi/\Phi_u = 0.5196$.

The transition between positive and negative slope at the inner radius takes place for some value of R_2/R_1 close to 3.8, but this limit has not been accurately determined yet.

Decreasing values of s_{max} lead to a wider outer domain of active constraint and a narrower inner domain. For increasing values of s_{max} the situation is reversed. For very high values of s_{max} the situation gets more complicated. It seems that the inner domain subdivides into three or

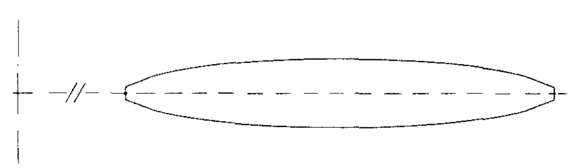

Fig. 1. Cross-section of optimal plate with $m = 0$, $R_1 = 100$, $R_2 = 101$, $s_{max} = 6$. $\Phi/\Phi_u = 0.7166$.

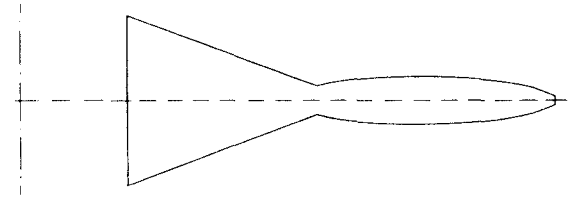

Fig. 2. Cross-section of optimal plate with $m = 0$, $R_1 = 0.25$, $R_2 = 1.25$, $s_{max} = 6$. $\Phi/\Phi_u = 0.5196$.

more subdomains with alternate negative and positive slope. However the solution has not been pursued further in this direction.

4. DOMAIN OF ACTIVE CONSTRAINT FOR $m > 0$

For the optimal plate the domain of active constraint (2.8) is U_s and we recognize that if U_s was known and given beforehand, the problem could be formulated in the following manner: Find a continuous function $h(r)$ subject to the volume restriction (2.7) and the condition

$$|h'(r)| = s_{max} \text{ for } r \in U_s \tag{4.1}$$

which minimizes the compliance Φ. This implies that we may disregard the condition (2.8) in the domain U_0. And this again implies that a thickness function which is subject to the conditions mentioned above but violates the constraint condition (2.8) anywhere in U_0 could not yield a smaller compliance than the minimum value found above.

With this in mind we can demonstrate that for the case $m > 1$ the domain U_0 is empty, i.e. the constraint (2.8) is active everywhere.

To show this, let us assume to the contrary that U_0 is not empty and that Φ_{min} with the corresponding thickness function $h_0(r)$ is the solution to the problem as formulated above and let $w_0(r)$ be the displacement function corresponding to $h_0(r)$.

Suppose $w_0 = 0$ everywhere in U_0.† Then $Q[\kappa] \equiv 0$ in this domain and therefore the constant Λ in (2.13) must vanish. Hence, the condition of optimality (2.13) is fulfilled for *any* $h(r)$ in U_0. However, since the total volume is given the optimal plate must have $h \equiv 0$ in U_0. Everywhere else, of course, $|h'(r)| = s_{max}$.

Having examined this rather pathological instance, we return to the general case, in which w_0 does not vanish everywhere in U_0. Under these circumstances, we may always find an interior point r_0 in U_0 and a small positive number ϵ, such that $w_0(r) \neq 0$ for all r satisfying the relation $(1-\epsilon^2)r_0 \leq r \leq (1+\epsilon^2)r_0$.

Let furthermore $h(r)$ be a continuous thickness function defined by

$$h(r) = \begin{cases} (1-\epsilon)h_0(r) & \text{for } R_1 \leq r \leq r_0(1-\epsilon^2) \\ (1-\epsilon)h_0(r) + H[r - r_0(1-\epsilon^2)] & \text{for } r_0(1-\epsilon^2) \leq r \leq r_0 \\ (1-\epsilon)h_0(r) + H[r_0(1+\epsilon^2) - r] & \text{for } r_0 \leq r \leq r_0(1+\epsilon^2) \\ (1-\epsilon)h_0(r) & \text{for } r_0(1+\epsilon^2) \leq r \leq R_2. \end{cases} \tag{4.2}$$

By a proper choice of H,

$$H = (R_2^2 - R_1^2)/r_0^2\epsilon^3 \tag{4.3}$$

we ensure that $h(r)$ satisfies the volume restriction (2.7). Thus $h(r)$ corresponds to a plate of thickness $(1-\epsilon)h_0(r)$ with a narrow wedge-shaped region at r_0.

The strain energy in the wedge-shaped region $r_0(1-\epsilon^2) \leq r \leq r_0(1+\epsilon^2)$ is given by the integral

$$W_\epsilon = \frac{\pi}{2}\int_{r_0-\epsilon^2}^{r_0+\epsilon^2} DQ[\kappa] r \, dr$$

for which we make the following estimate,

$$W_\epsilon \geq Q_{min}[\kappa]\frac{\pi}{2}\int_{r_0-\epsilon^2}^{r_0+\epsilon^2} Dr \, dr = Q_{min}[\kappa]\frac{A}{\epsilon} \tag{4.4}$$

where A is a number independent of ϵ. But the total strain energy (which is greater than W_ϵ)

†This could happen under very special load conditions. A necessary condition is $p(r) \equiv 0$ in U_0, but this is by no means sufficient.

equals the work done by external forces and is therefore bounded above. Hence, as $\epsilon \to 0$ we must have $Q_{min}[\kappa] \to 0$ and this can only happen if all three curvatures κ_{rr}, $\kappa_{\phi\phi}$, and $\kappa_{r\phi}$ tend to zero as $\epsilon \to 0$. Solving (2.3) for $\kappa_{rr} = \kappa_{\phi\phi} = \kappa_{r\phi} = 0$ we see that $w''(r_0)$, $w'(r_0)$ and $w(r_0)$ all must vanish when $\epsilon \to 0$.

From the expressions (4.2) it follows that $h \to h_0$ everywhere, except in the neighbourhood of r_0 where the plate acts as if it were clamped.

Since clamping at an interior circle r_0 must increase the stiffness of the plate, there are positive values of ϵ for which the compliance of the plate with thickness $h(r)$ according to (4.2) is *smaller* than the optimal value Φ_{min}.

Thus, the assumption that the domain U_0 is not empty leads to a contradiction and we must therefore conclude that the geometrical constraint (2.8) is active everywhere, i.e.

$$|h'(r)| = s_{max} \quad R_1 \le r \le R_2. \tag{4.5}$$

In the case $m = 1$ the solution of (2.3) with $\kappa_{rr} = \kappa_{r\phi} = \kappa_{\phi\phi} = 0$ will yield $w''(r_0) = 0$ and $w(r_0) = r_0 w'(r_0)$. This does not correspond to a clamping of the plate. However, it can be shown that by defining a thickness function $h(r)$ properly, w'' may be made to vanish at any finite number of equally spaced interior points of the domain U_0.

Thus in the limit the plate with thickness function $h(r)$ will behave like a rigid plate in U_0 and again that plate must be stiffer than the optimal plate $h_0(r)$. Therefore, the condition (4.5) holds for all $m > 0$.

5. THE CASE $m > 0$

We have found that in the case $m > 0$ the thickness $h(r)$ of the optimal plate is a piecewise linear function, which due to the requirement of continuity must have an alternating positive and negative slope $\pm s_{max}$ in successive intervals.

Let us assume that the intervals are separated by n radii $r_1, r_2, \ldots, r_n$ given in increasing order,

$$R_1 < r_1 < r_2 < \cdots < r_n < R_2. \tag{5.1}$$

With given values of $r_1, \ldots, r_n$ and a given sign of $h'(r)$ in—say—the first interval, the thickness $h(r)$ is uniquely determined by the volume restriction (2.7). The problem is therefore reduced to determining a finite set of numbers $r_1, \ldots, r_n$ and the sign of the slope in the first interval.

To solve this problem, assume that n is known and consider the compliance function $\Phi(r_1, \ldots, r_n)$ when $h'(r)$ is positive in the first interval, and

$$R_1 \le r_1 \le r_2 \le \cdots \le R_2. \tag{5.2}$$

To minimize Φ we shall employ a method of steepest decent using steps of equal length as long as the compliance is decreasing. Whenever a step has the effect of increasing the compliance the procedure is continued with a step length half the previous one. This method works well and the convergence is fast for small n, but slows down for larger numbers n.

Since the number of switching points n of the optimal plate is not known *a priori* we have to find this number by trial. If we try too small a value for n, say $p < n$ the value of Φ will improve (decrease) when p is increased. Too large a value of p will do no harm, since any two neighbouring switching points may coalesce. Also, if the slope $h'(r)$ of the optimal plate is negative in the first interval, contrary to our assumption, the procedure selected will make $r_1 \to R_1$ and with $r_1 = R_1$ the optimal solution will be obtained provided that p is large enough ($p \ge n + 1$).

Thus we have

$$\Phi_{opt} = \operatorname*{Min}_{r_1, \ldots, r_p} \Phi(r_1, \ldots, r_p) \quad p \ge n + 1. \tag{5.3}$$

To apply the method of steepest decent, we must first determine the gradient of Φ.

Assuming (5.1) to hold, let r_i be incremented by δr_i while all the remaining variables r_j $(j \neq i)$ are kept fixed.

From the geometry of the problem, and the condition that the total volume of the plate does not change, the increment δh due to δr_i is uniquely determined by

$$\delta h = \begin{cases} (2-1)^i s_{\max} \dfrac{R_2^2 - r_i^2}{R_2^2 - R_1^2} \delta r_i & \text{for} \quad R_1 \leq r < r_i \\ -2(-1)^i s_{\max} \dfrac{r_i^2 - R_1^2}{R_2^2 - R_1^2} \delta r_i & \text{for} \quad r_i < r \leq R_2. \end{cases} \tag{5.4}$$

The increment $\delta\Phi$ due to δh and therefore due to δr_i is found when δh according to (5.4) is substituted into (2.12). In the limit we have

$$\frac{\partial \Phi}{\partial r_i} = \frac{EC(-1)^i s_{\max}}{2(1-\nu^2)(R_2^2 - R_1^2)} \left[(r_i^2 - R_1^2) \int_{r_i}^{R_2} h^2 Q[\kappa] r \, \mathrm{d}r - (R_2^2 - r_i^2) \int_{R_1}^{r_i} h^2 Q[\kappa] r \, \mathrm{d}r \right]. \tag{5.5}$$

The gradient $(\partial\Phi/\partial r_i)$ is used to determine the direction of the steps in the numerical procedure. Thus

$$\delta r_i = -\delta r \frac{\partial \Phi}{\partial r_i} \Big/ \left| \frac{\partial \Phi}{\partial r_i} \right| \tag{5.6}$$

where δr is the step length and $|\partial\Phi/\partial r_i|$ the norm of the gradient. After each step the current value of Φ is determined. If Φ has increased, δr is replaced by $\delta r/2$.

For the optimal plate $(\partial\Phi/\partial r_i) = 0$ and we find the following necessary conditions of optimality

$$\frac{1}{r_{i+1}^2 - r_i^2} \int_{r_i}^{r_{i+1}} h^2 Q[\kappa] r \, \mathrm{d}r = \frac{1}{R_2^2 - R_1^2} \int_{R_1}^{R_2} h^2 Q[\kappa] r \, \mathrm{d}r \quad i = 0, \ldots, n \tag{5.7}$$

where $r_0 = R_1$ and $r_{n+1} = R_2$.

The quantity $h^2 Q$ is proportional to the mean strain energy density per unit thickness of the plate. Equation (5.7) states that this mean strain energy density averaged over any full interval of constant slope equals the average taken over the whole plate. The conditions are nothing but a discretized version of the optimality condition (2.13), which states that this mean energy density is uniformly distributed in any domain in which the constraint (2.7) is not active.

From the results obtained, three examples are given below. A simply supported plate with $R_2/R_1 = 5$ was optimized for the case $m = 4$ with three different constraints on the maximum slope, $s_{\max} = 2$ (Fig. 3), $s_{\max} = 4$ (Fig. 4), and $s_{\max} = 8$ (Fig. 5). The number of intervals of constant slope was found to be 3, 5, and 9 respectively. In all three cases the optimality conditions (5.7) were satisfied to a high degree of accuracy. The required accuracy was obtained after 110 iterations in the first case ($s_{\max} = 2$), after 793 iterations in the second case ($s_{\max} = 4$) and after 7890 iterations in the last case ($s_{\max} = 8$). It was found that convergence was speeded up when the step-length was increased by a factor 1.05 after each step that resulted in an improved value of Φ and decreased to half the previous step whenever Φ increased.

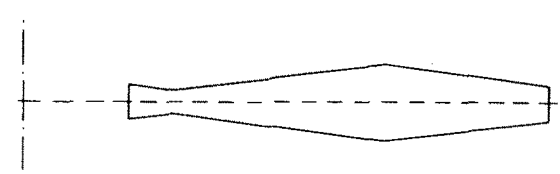

Fig. 3. Cross-section of optimal plate with $m = 4$, $R_1 = 0.25$, $R_2 = 1.25$, $s_{\max} = 2$, $\Phi/\Phi_u = 0.8934$.

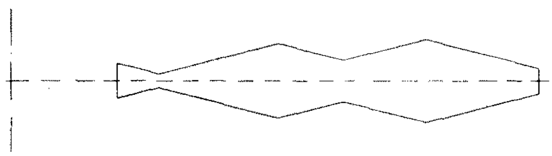

Fig. 4. Cross-section of optimal plate with $m = 4$, $R_1 = 0.25$, $R_2 = 1.25$, $s_{max} = 4$. $\Phi/\Phi_u = 0.8502$.

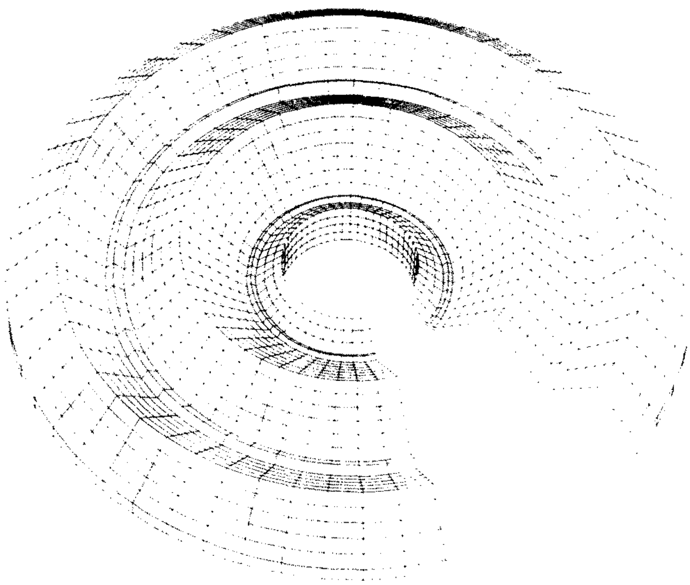

Fig. 5. Exploded view of optimal plate with $m = 4$, $R_1 = 0.25$, $R_2 = 1.25$, $s_{max} = 8$. $\Phi/\Phi_u = 0.7768$.

6. DISCUSSION

A constraint on the slope $h'(r)$ has a profound effect on the problem of designing optimal plates. Of prime importance is the fact that the problem becomes well-posed. In addition an unexpected result is that except for a limited range of the parameters in the special case of rotational symmetry, the constraint on the slope is active everywhere. The optimal plate (assumed to be axisymmetric) will have a cross-section bounded by straight lines for all values of $m > 0$ and all finite values of s_{max}.

From the requirement of given volume it follows that the number of wedges n must increase with s_{max} so that $n \to \infty$ as $s_{max} \to \infty$. This explains why the problem becomes ill-posed when there is no restriction on the slope $h'(r)$.

The range of values for all parameters characterizing a problem is wide. Only a few cases, believed to be typical, have been treated and certainly the problem could and should be pursued in a number of different directions.

From a theoretical, although perhaps not from a practical point of view the behaviour of the solution for $m > 0$ and large values of s_{max} would be interesting to follow. But there are indications that this direction might lead to numerical difficulties.

Of considerable interest is certainly the problem of optimizing plates of more general shapes, starting perhaps with rectangular plates. A reasonable conjecture is that a constraint on the gradient of the thickness will result in well-posed problems with solutions characterized by $|\text{grad } h| = s_{max}$.

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